COMBINATORICA
Akadémiai Kiadó — Springer-Verlag

COMBINATORICA 10 (1) (1990) 107—109

NOTE

ON RECOGNIZING INTEGER POLYHEDRA

C. H. PAPADIMITRIOU[1] and M. YANNAKAKIS

Received September 9, 1988

We give a simple proof that, determining whether a convex polytope has a fractional vertex, is NP-complete.

Schrijver notes in his book [1] that the following problem

"Given a rational system $Ax \leqq b$ of linear inequalities, does it determine an integer polyhedron?"

is in coNP (that is, its complement of asking whether a polyhedron has a fractional vertex is in NP). He asks whether the integrality problem is also in NP, or even in P (it *is* in P for fixed dimensions). It would be rather remarkable if the answer to the latter question were positive, since it would imply that one can recognize integer programs that are amenable to solution by linear programming techniques, independently of the cost vector. Many important classes of linear programs, such as totally unimodular, balanced, and others, see [1], are weak sufficient conditions for integrality. In other words, many of the important results and open questions in polyhedral combinatorics are in effect special cases of this general question. In this note we show that the problem of testing whether a polyhedron is integral is coNP-complete. The proof is quite easy, and follows.

We consider a directed graph $G=(V, A)$ with $|V|=n$ nodes, where n is a *prime* number (we shall somehow use this later). We create a new bipartite directed graph $G'=(V\cup V', A')$, where $V'=\{v': v\in V\}$ is another copy of V, and A' contains all arcs $A_1'=\{(v, v'): v\in V\}$ and also all arcs $A_0'=\{(u', v): (u, v)\in A\}$. Notice already that there is a one-to-one correspondence between paths in G' and paths in G.

Consider now the node-arc incidence matrix $A_{G'}$ of this graph, with the (linearly redundant) last row omitted, as usual, and the linear program LP:

$$A_{G'}x = 0$$

$$\sum_{e\in A_1'} x_e = (n-1)!$$

$$x \geqq 0.$$

What are the basic feasible solutions of LP? Their bases are obtained by taking $2n$ columns of $A_{G'}$, and complementing them with a $2n$th row of zeroes and ones, depending on whether the column corresponds to an arc in A_0' or A_1'. Now, the

AMS subject classification (1980): 68Q25

[1] Research supported by the National Science Foundation.

determinant of this matrix can be calculated by expanding along the last row. Note, however, that the last row contains at most n ones. Furthermore, the matrix except for the last row is totally unimodular [1] and thus all minors have determinants that are integers bounded in absolute value by one. *It follows that the determinants of all bases of LP are integers bounded in absolute value by n.*

If a basis has determinant $n-1$ or less, then the corresponding basic feasible solution is integer (as all integers below $n-1$ divide the right-hand side). Thus, LP can possibly have a fractional solution only if there is a basis with determinant n.

What kind of basis of LP has determinant n? Each basis B contains $2n$ columns, and can therefore be viewed as a subgraph of G' also denoted B. B must first contain all arcs in A_1' (since they correspond to the only n columns with a 1 in the last row). Furthermore, any submatrix of B that result if we delete the last row and any arc of A_1' should have a nonzero determinant. This means that, if we delete any arc in A_1', the remaining part of B should be weakly connected (i.e., connected if we disregard directions).

Recall that G' is a directed graph with $2n$ nodes. We claim that any subgraph B of G' with $2n$ arcs that has the property that, deleting any single arc in A_1' leaves B connected is a *weak Hamilton cycle* of G', that is, a subgraph whose underlying undirected graph is a simple cycle visiting all nodes. In proof, just notice that the underlying graph of B must be a spanning tree plus an edge, and it can have no node of degree one. Because the edge incident upon such a node can be neither an A_1' edge (its deletion would disconnect the graph) neither an A_0' edge (the A_1' arc corresponding to the node would be missing, contrary to our previous observation). However, every other arc on the weak cycle belongs to A_1', and, if $(u, v)\in A_1'$, then there is no other arc coming into v or leaving u. Hence, the weak Hamilton cycle B is in fact a Hamilton cycle (all arcs are in the same direction). Thus, the only possible bases of LP that have determinant n are the Hamilton cycles of G'. One direction of the reduction has been proved: If LP has a fractional solution, then G' has a Hamilton cycle.

Conversely, it is easy to see that the columns corresponding to a Hamilton cycle form a basis, and the basic feasible solution has $\frac{(n-1)!}{n}$ on all arcs. Since n is a prime, it is a fractional basic feasible solution. We have shown the following:

Lemma 1. *LP has a fractional basic feasible solution if and only if G' has a Hamilton cycle.* ∎

It is easy to see, however, that G' has a Hamilton cycle if and only if the original directed graph G has one, since the cycles of G and G' are in the one-to-one correspondance $(v_1, v_2, \ldots, v_n)\leftrightarrow(v_1, v_1', v_2, v_2', \ldots, v_n')$. Now G was a perfectly arbitrary graph, except for the requirement that it have a prime number of nodes. The following is easy to show:

Lemma 2. *The Hamilton cycle problem remains NP-complete even when restricted to directed graphs with a prime number of nodes.*

Sketch: Replace a node in the original graph with a chain of nodes, enough to make the total number of nodes a prime; recall that there is always a prime between n and $2n$. ∎

From the two Lemmata we have our result:

Theorem. *Telling whether a given set of linear inequalities describes an integer polyhedron is coNP-complete.* ▮

Notice that in our construction it is not necessary to use large integers, since we can define $(n-1)!$ by a sequence of equations, using small numbers: Introduce variables $y_1 \dots y_{n-1}$, and the equations $y_1=1$ and $y_k=ky_{k-1}$, $k=2, \dots, n-1$. It follows that the fractional vertex problem is *strongly NP-complete;* that is, it is NP-complete even if the integers involved are restricted to be at most polynomially large in the size of the input (many thanks to Peter Shor for pointing this out to us).

There are several related problems that remain open. First, we note that we still do not have a polynomial time algorithm for recognizing balanced matrices. In contrast, we conjecture that telling whether a matrix has subdeterminants that are either 0, ± 1, or ± 2, is NP-complete (this is obviously a generalization of total unimodularity, and our Theorem would follow from such an NP-completeness result). Further related important open complexity questions are the recognition problem of totally dual integral linear programs, and the related problem of recognizing whether a set of integer vectors form a Hilbert base (see [1]).

References

[1] A. Schrijver, *Theory of Linear and Integer Programming*, Wiley-Interscience, 1986.

Christos H. Papadimitriou

Dep. of Computer Science
and Engineering,
Univ. of California at San Diego

Mihalis Yannakakis

AT & T Bell Laboratories,
Muray Hill, New Jersey, USA